The basic knowledge necessary to solve a differential equation

One of the most fundamental types of differential equations is a linear equation with constant coefficients. A linear equation consists only of linear terms, which may include constant coefficients. Transitioning to a homogeneous equation is required - we move all terms involving the function y and its derivatives, then equate them to zero, obtaining a homogeneous equation. Next, it is necessary to treat the constant as a function dependent on x. We calculate the derivatives of both sides and utilize the results in the initial equation. The result is the sum of solutions from both parts.

We consider the following differential equation:

$$y' - e^{2x} - ye^{x} = 0 \tag{1}$$

Where:

$$y(x) \tag{2}$$

Is our encountered function.

Transition to a homogeneous equation

We move all terms involving the function y and its derivatives to one side and equate it to zero, obtaining a homogeneous equation.

The solution to the resulting separated variable equation

The obtained homogeneous equation is a separated variable equation and takes the form:

$$y(x)e^{x} + \frac{d}{dx}y(x) = 0 \tag{3}$$

We consider the following differential equation:

$$y' - ye^{x} = 0 \tag{4}$$

Where:

$$y(x) \tag{5}$$

Is our encountered function.

Separation of variables

If possible, we perform the commonly known separation of variables x and y. By separating variables, we mean separating terms containing the variable x from those containing the variable y. To accomplish this separation, we can assign these terms to functions $h(x)$ and $g(x)$, as shown below:

$$g(x) = e^{x} \tag{6}$$

$$h(x) = y \tag{7}$$

Apart from these elements, our equation also contains the term y', which can also be written as:

$$y' = \frac{dy}{dx} \tag{8}$$

After substituting this expression, it can be observed that in subsequent steps, multiplying the equation on both sides by dx enables integrating both sides with respect to different variables.

Integration of separated variables

$$\frac{dy}{dx} = ye^x \tag{9}$$

To do this, we multiply both sides by dx and move $y(x)$ to the opposite side of the equation:

$$\frac{dy}{y} = e^x dx \tag{10}$$

Then, to "go back" to the original function and obtain the result in the form of the f y, we integrate both sides of the obtained equation. Importantly, when integrating, it's essential to remember to add the constant of integration in any form, typically denoted as C or D.

$$\int \frac{1}{y}\,dy = \int e^x\,dx \tag{11}$$

After integration, we obtain the following result:

$$\log(y) = e^x + D \tag{12}$$

Simplification of the solution obtained after integrating the equations

After rearrangment a solution has the following form:

$$y(x) = C_1 e^{e^x} \tag{13}$$

The obtained expression represents the solution to a differential equation with separated variables. The constant of integration can be determined when initial conditions are provided in the problem, i.e., the value of the function for a specific x.

Variation of constant

The next step is to treat the constant as a function dependent on x:

$$y = -1 - e^x + \mathrm{C}_1(x)e^{e^x} \tag{14}$$

Next, you need to calculate the derivatives of both sides of the equation.Upon obtaining the equation in the following form:

$$y' = -e^x + \mathrm{C'}(x)e^{e^x} + \mathrm{C}_1(x)e^x e^{e^x} \tag{15}$$

Thanks to this operation, we can now substitute y and y' into the original equation.

$$e^{2x} - y(x)e^x + \frac{d}{dx}y(x) = 0 \tag{16}$$

And obtain:

$$e^x - e^{2x} + \mathrm{C'}(x)e^{e^x} - \left(\mathrm{C}_1(x)e^{e^x} - e^x - 1\right)e^x + \mathrm{C}_1(x)e^x e^{e^x} = 0 \tag{17}$$

Then, if everything has been calculated correctly, the values of $C(x)$ should simplify to a form from which we can easily determine $C'(x)$, obtaining:

$$\mathrm{C'}(x) = 0 \tag{18}$$

After integrating both sides, we obtain the value of the constant:

$$\mathrm{C}_1(x) = 0 \tag{19}$$

To obtain the final result, we need to substitute the obtained constant into the first equation where we introduced the constant as a variable:

$$y = -1 - e^x \tag{20}$$

