The basic knowledge necessary to solve a differential equation:

The Bernoulli differential equation is a special type of first-order ordinary differential equation. Its general form is:

$$y' + P(x)y = Q(x)y^n \tag{1}$$

Where y is the unknown function dependent on the variable x, $P(x)$ and $Q(x)$ are given functions dependent on x, and n is a constant real number, different from 0 and 1. Characteristic of Bernoulli equations is that they are nonlinear due to the presence of the term y^n on one side of the equation. Solutions to such equations can be challenging to obtain in general, but for certain values of n, certain techniques can be applied to simplify the solution. In the case of $n = 0$, the Bernoulli equation becomes linear, and for $n = 1$, it can be transformed into the form of a linear ordinary differential equation of the first order. In other cases, special techniques such as variable substitution or order reduction may be necessary to obtain the general solution.

We consider the following differential equation:

$$y + xy' + xy^2 = 0 \tag{2}$$

Where:

$$y(x) \tag{3}$$

Is our encountered function.

Determining the new function z dependent on x in order to transition to a linear equation.

To make the substitution, it is necessary to define n by finding the highest power of y, where n is the exponent of this function.

$$n = 2 \tag{4}$$

Next, you need to make the substitution and transition to the new function z, using the following formula:

$$z = y^{1-n} \tag{5}$$

In our case:

$$z = \frac{1}{y} \tag{6}$$

The next step is to determine the first derivative of the function z:

$$z' = \frac{(1-n)\, y^{1-n}(x) \frac{d}{dx} y(x)}{y(x)} \tag{7}$$

After substituting the n value :

$$z' = -\frac{\frac{d}{dx} y(x)}{y^2(x)} \tag{8}$$

Next, we need to determine y and y' from the above equations.

$$y = \frac{1}{z(x)} \tag{9}$$

$$y' = \frac{\frac{d}{dx} z(x)}{z^2(x)} \tag{10}$$

Transition to a linear equation

The last step in transitioning to a linear equation is substituting the previously calculated y and y' into the original equation:

$$x - z(x) + x\frac{d}{dx}z(x) = 0 \tag{11}$$

We compute the obtained equation in the following manner:

We consider the following differential equation:

$$x - z + xz' = 0 \tag{12}$$

Where:

$$z(x) \tag{13}$$

Is our encountered function.

Transition to a homogeneous equation

We move all terms involving the function y and its derivatives to one side and equate it to zero, obtaining a homogeneous equation.

The solution to the resulting separated variable equation

The obtained homogeneous equation is a separated variable equation and takes the form:

$$z(x) + x\frac{d}{dx}z(x) = 0 \tag{14}$$

We consider the following differential equation:

$$z + xz' = 0 \tag{15}$$

Where:

$$z(x) \tag{16}$$

Is our encountered function.

Separation of variables

If possible, we perform the commonly known separation of variables x and y. By separating variables, we mean separating terms containing the variable x from those containing the variable y. To accomplish this separation, we can assign these terms to functions $h(x)$ and $g(x)$, as shown below:

$$g(x) = \frac{1}{x} \tag{17}$$

$$h(x) = z \tag{18}$$

Apart from these elements, our equation also contains the term z', which can also be written as:

$$z' = \frac{dz}{dx} \tag{19}$$

After substituting this expression, it can be observed that in subsequent steps, multiplying the equation on both sides by dx enables integrating both sides with respect to different variables.

Integration of separated variables

$$\frac{dz}{dx} = \frac{z}{x} \tag{20}$$

To do this, we multiply both sides by dx and move $z(x)$ to the opposite side of the equation:

$$\frac{dz}{z} = \frac{1}{x}dx \tag{21}$$

Then, to "go back" to the original function and obtain the result in the form of the f y, we integrate both sides of the obtained equation. Importantly, when integrating, it's essential to remember to add the constant of integration in any form, typically denoted as C or D.

$$\int \frac{1}{z}\, dy = \int \frac{1}{x}\, dx \tag{22}$$

After integration, we obtain the following result:

$$\frac{y}{z} = \log(x) + D \tag{23}$$

Simplification of the solution obtained after integrating the equations

After rearrangment a solution has the following form:

$$z(x) = C_1 x \tag{24}$$

The obtained expression represents the solution to a differential equation with separated variables. The constant of integration can be determined when initial conditions are provided in the problem, i.e., the value of the function for a specific x.

Variation of constant

The next step is to treat the constant as a function dependent on x:

$$y = x\left(\mathrm{C}_1(x) - \log(x)\right) \tag{25}$$

Next, you need to calculate the derivatives of both sides of the equation.Upon obtaining the equation in the following form:

$$y' = -\log(x) + x\left(\mathrm{C'}(x) - \frac{1}{x}\right) + \mathrm{C}_1(x) \tag{26}$$

Thanks to this operation, we can now substitute y and y' into the original equation.

$$x - z(x) + x\frac{d}{dx}z(x) = 0 \tag{27}$$

And obtain:

$$x + x\left(x\left(\mathrm{C'}(x) - \frac{1}{x}\right) + \mathrm{C}_1(x) - \log(x)\right) - x\left(\mathrm{C}_1(x) - \log(x)\right) = 0 \tag{28}$$

Then, if everything has been calculated correctly, the values of $C(x)$ should simplify to a form from which we can easily determine $C'(x)$, obtaining:

$$\mathrm{C'}(x) = 0 \tag{29}$$

After integrating both sides, we obtain the value of the constant:

$$\mathrm{C}_1(x) = 0 \tag{30}$$

To obtain the final result, we need to substitute the obtained constant into the first equation where we introduced the constant as a variable:

$$y = -x\log(x) \tag{31}$$